Analytic functions

Def. f is (complex) differentiable at z_0 if
$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} =: f'(z_0) \text{ exists.}$$

Important Where is f defined? For this to be interesting, should be defined in $B(z_0, r)$ for some $r>0$.

Equivalent definition: $f(z) = f(z_0) + (z-z_0)\varphi(z)$, where $\varphi(z)$ continuous at z_0, $\varphi(z_0) = f'(z_0)$.

Proof (of equivalency) $(\Uparrow)$ $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} = \lim_{z\to z_0}\varphi(z) = f'(z_0)$

$(\Downarrow)$ Take $\varphi(z) := \begin{cases}\frac{f(z)-f(z_0)}{z-z_0}, & z\neq z_0\\ f'(z_0), & z = z_0\end{cases}$.

Remark. Differentiability at one point is not interesting.

$F(z) = \Re z|\Re z|$ is complex differeniable for all z with $\Re z = 0$, but not twice differentiable for those z.

Interesting: differentiability at every point of some $B(z_0,\delta)$ - some neighborhood of z_0.

Def f is called ***analytic*** (or ***holomorphic***) at a point $z_0 \in \mathbb{C}$ if for some $\delta > 0$ it has a derivative at *every* $z \in B(z_0, \delta)$.

Thm. (the same as in Calculus).

1) If $f'(z)$, $g'(z)$ exist, then
$(f\pm g)'(z) = f'(z) \pm g'(z)$ - exist.
$(fg)'(z) = f'(z)g(z) + f(z)g'(z)$ - exist
$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}$ if $g(z)\neq 0$

2) If $f'(z)$ exist, and g' exist at $f(z)$, then
$(g(f(z)))' = g'(f(z))\cdot f'(z)$ - exist (Chain Rule).

Proof The same as in Calculus! ∎

Example 0. $f(z) \equiv c$. $f'(z) \equiv 0$.

Example 1 $f(z)=z$, $f'(z_0)=\lim_{z\to z_0}\frac{z-z_0}{z-z_0}=1.$

Example 1: Non-differentiable: $f(z)=\overline{z}$.

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=\lim_{z\to z_0}\frac{\overline{z-z_0}}{z-z_0}=\lim_{h\to 0}\frac{\overline{h}}{h}$$ – does not exist!

Real and complex differentiability.

Complex: $\lim_{h\to 0}\frac{|f(z+h)-f(z)-f'(z)h|}{|h|}=0$

Real: $\lim_{h\to 0}\frac{|f(z+h)-f(z)-T(h)|}{h}=0$ $\quad f=u+iv$

$T(h):\mathbb{R}^2\to\mathbb{R}^2$ – linear map. $\quad h=x+iy=\begin{pmatrix}x\\y\end{pmatrix}$

From calculus, $T(h)=\frac{\partial f}{\partial x}x+\frac{\partial f}{\partial y}y=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)x+\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)y$ – in complex form

$T(h)=\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$ – in real form.

Real notation:

$$\begin{pmatrix}u(x_0+x,y_0+y)\\ v(x_0+x,y_0+y)\end{pmatrix}=\begin{pmatrix}u(x_0,y_0)\\ v(x_0,y_0)\end{pmatrix}+T\begin{pmatrix}x\\y\end{pmatrix}+\overline{o}\left(\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|\right)$$

$$\frac{\partial f}{\partial x}(z):=\lim_{x\to 0}\frac{f(z+x)-f(z)}{x}$$

$$\frac{\partial f}{\partial y}(z):=\lim_{y\to 0}\frac{f(z+iy)-f(z)}{y}.$$

$x=\frac{h+\overline{h}}{2}$, $\quad y=\frac{h-\overline{h}}{2i}$.

$$T(h)=\frac{\partial f}{\partial x}\frac{h+\overline{h}}{2}-\frac{i}{2}\frac{\partial f}{\partial y}(h-\overline{h})=\frac{1}{2}\left(\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}\right)h+$$
$$+\frac{1}{2}\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right)\overline{h}.$$

Notation: $\frac{\partial f}{\partial z}:=\frac{1}{2}\left(\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}\right)$ $\quad \frac{\partial f}{\partial \overline{z}}=\frac{1}{2}\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right)$

When is real differentiable function complex differentiable?

$\lim \quad f(z+h)-f(z) \quad T(h) \quad \partial f \; h \quad \partial f \; \overline{h}$

When is real differentiable function complex differentiable?

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\frac{T(h)}{h} = \lim_{h\to 0}\frac{\partial f}{\partial z}\cdot\frac{h}{h} + \lim_{h\to 0}\frac{\partial f}{\partial \bar z}\,\frac{\bar h}{h}$$

So, since $\lim_{h\to 0}\frac{\bar h}{h}$ — does not exist, we get.

Theorem (Cauchy-Riemann) Let f be a real-differentiable function at z_0. It is complex differentiable if and only if $\frac{\partial f}{\partial \bar z}(z_0) = 0$.

Augustin-Louis Cauchy

Bernhard Riemann

Remark: $f'(z) = \frac{\partial f}{\partial z}$ in this case.

Other form: $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = 0$. Or

$$\begin{pmatrix} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{pmatrix}$$ — Cauchy-Riemann equations.

Matrix of $T(h)$: $\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix} = M_{f'(z)}$.

$$\boxed{f'(z): \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}}$$

A Real Analysis refresher:

Suppose that the function $f = u + iv$: $\boldsymbol{R}^2 \to \boldsymbol{R}^2$ has all the first partial derivatives for all points $z \in B(z_0, r)$.
What can you say about f?

1. f is continuous in $B(z_0, r)$, but not always differentiable.
2. f is differentiable in $B(z_0, r)$.
3. The functions u and v are differentiable in $B(z_0, r)$.
4. f can be discontinuous at some points of $B(z_0, r)$.

$z = x+iy$

$$f(z) = \begin{cases} \frac{xy}{x^2+y^2} + i0, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}\Big|_0 = 0$.

Not continuous at 0: $z = \varepsilon(1+i)$ $f(z) = \frac{\varepsilon^2}{2\varepsilon^2} = \frac{1}{2} \not\to 0$

Theorem (Multivariable Calculus). Let $u(x,y)$ has pariatial derivatives $\frac{\partial u}{\partial x}(x,y)$ and $\frac{\partial u}{\partial y}(x,y)$ for all $(x,y) \in B\left((x_0,y_0),\delta\right)$ which are ***continuous*** at (x_0,y_0). Then u is differentiable at (x_0,y_0).

Remark Need to assume real-differentiability apriori.
$\exists f : \frac{\partial f}{\partial z}, \frac{\partial f}{\partial \bar z}$ - exist every where, $\frac{\partial f}{\partial \bar z} \equiv 0$, yet f is not everywhere analytic! $\left(f(z) = \begin{cases} e^{-\frac{1}{z^4}}, & z \neq 0 \\ 0, & z = 0 \end{cases} \right)$

Thm (Looman-Menchoff) If $f = u + iv$ is continuous $\forall z \in B(z_0, r)$, all the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist $\forall z \in B(z_0, r)$, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Then f is analytic in $B(z_0, r)$.

Theorem Let f be analytic in $B(z_0,\delta)$, $f'(z) \equiv 0 \; \forall z \in B(z_0,\delta)$.
Then $f(z) \equiv$ const.

Proof. $f'(z) = 0 \Rightarrow \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \equiv 0 \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} \equiv 0 \overset{\text{Calculus}}{\Rightarrow}$
$u \equiv$ const, $v \equiv$ const ∎

Remark Can assume less: $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \equiv 0$ (without assuming differentiability apriori).
Differentiability follows from continuity.

Proof. $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ - continuous $\overset{\text{Calculus}}{\Rightarrow}$ f is real-differentiable
$\frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0 \Rightarrow f$ is analytic ∎